Compressible Potential Flow:
The Full Potential Equation
Introduction

• Recall that for incompressible flow conditions, velocity is not large enough to cause density changes, so density is known. Thus the unknowns are velocity and pressure. We need two equations. The Continuity equation is enough to solve for velocity as a function of time and space.

• Pressure can be obtained from the Bernoulli equation, which comes from Momentum Conservation.

• If the flow is irrotational as well, we can define a potential, and reduce the continuity equation to the form of the Laplace equation.

\[ \nabla^2 \Phi = 0 \]

Here \( \Phi \) is defined such that

\[ \vec{U} = \nabla \Phi \]

\[ u = \frac{\partial \Phi}{\partial x} \quad v = \frac{\partial \Phi}{\partial y} \quad w = \frac{\partial \Phi}{\partial z} \]
High-speed flows can also be irrotational.

**Objectives**
1. Derive the potential equation for compressible flow.
2. Reduce to linearized form for small-perturbation analysis.
3. Apply results to thin airfoils and slender axisymmetric bodies.

**Note**
Generally, flight vehicles are designed to create as little "perturbation" as possible, because large disturbances cause large drag.

We will derive the potential equation for 2-D flows. At the end, it will be obvious how to extend it to 3-D flow, so we will just write down the 3-D equation.

**Assume:**
1. Isentropic flow.
2. Steady flow.
Continuity Equation

\[ \frac{D\rho}{Dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]

Here, as we recall

\[ \frac{D(\alpha)}{Dt} = \frac{\partial \alpha}{\partial t} + u \frac{\partial \alpha}{\partial x} + v \frac{\partial \alpha}{\partial y} + w \frac{\partial \alpha}{\partial z} \]

Steady:

\[ \frac{\partial \alpha}{\partial t} = 0 \]

\[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \rightarrow (1) \]
Momentum Conservation

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \rightarrow (2) \]

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \]
Speed of sound

\[ a^2 = \frac{\partial p}{\partial \rho} \bigg|_s \rightarrow (3) \]

i.e.

\[ \frac{\partial p}{\partial y} = \frac{\partial p}{\partial \rho} \bigg|_s \frac{\partial \rho}{\partial y} = a^2 \frac{\partial \rho}{\partial y} \]

\[ \frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho} \bigg|_s \frac{\partial \rho}{\partial x} = a^2 \frac{\partial \rho}{\partial x} \rightarrow (3a) \]

Substitute (3a) in (2),

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{a^2}{\rho} \frac{\partial p}{\partial x} \rightarrow (4a) \]

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{a^2}{\rho} \frac{\partial p}{\partial y} \rightarrow (4b) \]
For homework,

1. Multiply (4a) by $u$, and (4b) by $v$ and add. Call this eqn (5)

2. Expand eqn (1) and substitute in eqn (5)

Solution:

\[
\left(\frac{u^2}{a^2} - 1\right) \frac{\partial u}{\partial x} + \frac{uv}{a^2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \left(\frac{v^2}{a^2} - 1\right) \frac{\partial v}{\partial y} = 0 \rightarrow (6)
\]

This equation contains derivatives of $u$ and $v$. If we could define a potential, we could reduce the number of variables.
Assume Irrotational Flow

$$\nabla \times \vec{U} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

If this is true, then a potential $\Phi$ can be defined such that

$$\vec{U} = \nabla \Phi$$

$$u = \frac{\partial \Phi}{\partial x}$$

$$v = \frac{\partial \Phi}{\partial y}$$

$$|\vec{U}|^2 = u^2 + v^2$$

(7)
Substituting in eqn. (6) or using the notation

\[ \phi_x \equiv \frac{\partial \phi}{\partial x}, \phi_y \equiv \frac{\partial \phi}{\partial y} \]

\[ \left( \frac{u^2}{a^2} - 1 \right) \phi_{xx} + \frac{2uv}{a^2} \phi_{xy} + \left( \frac{v^2}{a^2} - 1 \right) \phi_{yy} = 0 \rightarrow (8) \]

What is ‘a’, the local speed of sound? To see this, go to the energy equation for steady adiabatic flow.
\[ h + \frac{1}{2} (u^2 + v^2) = \text{const} = h_0 \rightarrow (9) \]

\[ h = c_p T = \frac{R \gamma}{\gamma - 1} T \rightarrow (10) \]

\[ \Rightarrow \gamma R T + \frac{\gamma - 1}{2} (u^2 + v^2) = \text{const} \]

or

\[ \alpha^2 + \frac{\gamma - 1}{2} V^2 = \text{const} = \alpha_\infty^2 + \frac{\gamma - 1}{2} V_\infty^2 = \alpha_0^2 \]
Thus,

\[ \alpha^2 = \alpha_0^2 - \frac{\gamma - 1}{2} V^2 \rightarrow (11) \]

or,

\[ \alpha^2 = \alpha_0^2 - \frac{\gamma - 1}{2} (\phi_x^2 + \phi_y^2) \]

**Extension to 3-D flow**

By inspection, equations (8) and (11) can be written, for 3-D potential flow, as:

\[
\left( \frac{u^2}{a^2} - 1 \right) \phi_{xx} + \left( \frac{v^2}{a^2} - 1 \right) \phi_{yy} + \left( \frac{w^2}{a^2} - 1 \right) \phi_{zz} + \frac{2}{a^2} \left[ uv \phi_{xy} + vw \phi_{yz} + uw \phi_{zx} \right] = 0 \rightarrow (12)
\]

**Eq. 12 is the 3D Steady Full Potential Equation**

where,

\[ \alpha^2 = \alpha_0^2 - \frac{\gamma - 1}{2} \left[ \phi_x^2 + \phi_y^2 + \phi_z^2 \right] \rightarrow (13) \]
Note:

1. This is still an exact equation. We have not made any approximations. In other words, if the flow satisfies our assumptions (steady, irrotational, isentropic), the solution of this will still give exact results. Note the very important distinctions between this and the approximate equations that we will presently derive based on "engineering estimates".

2. We have not made the assumption that disturbances are small; although we have implicitly assumed that the shocks are absent (or quite weak). Thus, this equation can be used to calculate transonic flows over very complex configuration, such as fighter aircraft and rotor blades.

3. This equation is highly non-linear in $\Phi$.

4. Superposition of solutions will not work.

5. Equally valid for subsonic and supersonic flows.

6. We have used $h = c_p T$, and $\gamma = const$

These can be easily modified, and the resulting equations can be used to analyze high-temperature flows. (rocket exhaust nozzles)
Classic Linear Supersonic Aerodynamic Design Process

US SST and Pre 1990

Prior Knowledge
- Design Mission Objectives
- Configuration Features
- Initial Component Sizes

Linear Design Loop

Linear Theory:
- Design integration
- Design Optimization

Validate Performance?

NO

Satisfy Design Constraints?

NO

Apply Real Flow Design Constraints

YES

Linear Theory Cp Calculation

Wind Tunnel Test Programs
- Validate Supersonic Performance
- Off Design Flap Optimization

YES

Model to Airplane Geometry Differences
- Excrecence / Miscellaneous Drag
- Power Effects
- Scale Viscous Drag to Full Scale Conditions

Using Flat Plate Skin Friction Theory

Full Scale Performance

Courtesy Dr. B. Kulfan, BOEING COMPANY, 2004
Refined Linear Supersonic Aerodynamic Design Process

Initial HSCT Technology – 1990

Prior Knowledge
• Design Mission Objectives
• Configuration Features
• Initial Component Sizes
• Wing Leading Edge Design

Linear Theory:
• Design integration
• Design Optimization

LINEAR DESIGN

YES

Satisfy Design Constraints?

NO

Wing Leading Edge Design

CFD Analyses
• Inviscid / Viscous
• Apply Real Flow Design Constraints?
• Off design Flap Parametric Optimization

YES

Wind Tunnel Test Programs
• Validate Supersonic Performance
• Off Design Flap Optimization

NO

Validate Performance?

NO

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Current Non-Linear Supersonic Aerodynamic Design Process

HSR Technology - 1998

Prior Knowledge
- Design Mission Objectives
- Configuration Features
- Initial Component Sizes
- Wing Leading Edge Design

Linear Theory:
- Design integration
- Design Optimization

LINEAR DESIGN

CFD Analyses
- Inviscid Analyses
- Viscous Analyses
- Viscous Determination of the flow characteristics

Non-Linear Design Optimization
- Point Design
- Off Design Flap Optimization
- Multi-Point Optimization

Wind Tunnel Test Programs
- Validate Supersonic Performance
- Off Design Flap Optimization

Validate Performance?

Satisfy Design Constraints?

NO

YES

Full Scale Performance

• Model to Airplane Geometry Differences
• Excrecence / Miscellaneous Drag
• Power Effects
• Scale Viscous Drag
• Scale Pressure Drag?